Rigid Body Dynamics (Unconstrained)

Particles:
State vector of a particle \( \mathbf{y}_t = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \)

State vector of \( N \) particles
\[
\mathbf{y}_t = \begin{pmatrix}
  x_1(t) \\
  v_1(t) \\
  \vdots \\
  x_n(t) \\
  v_n(t)
\end{pmatrix}
\]

To simulate the motion of these particles, we need to know the force acting on them.

Change in \( \mathbf{y}_t \) over time is
\[
\frac{d}{dt} \mathbf{y}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix} \quad \text{For a single particle.}
\]

Easily extended for \( n \) particles. If we know how to simulate a single particle, we can simulate \( n \) particles.
Rigid Body

The rigid body has both position and orientation.

represent position of a rigid body using a translation \( x(t) \).

How to represent the orientation of a rigid body?

\( R(t) \in \mathbb{R}^{3 \times 3} \), a rotation matrix.

\( x(t) \) and \( R(t) \) are referred to as the spatial variables of the rigid body.

Body space.

World Space.

\( p(t) \) is point \( p_0 \) expressed in the world space.

\[ p(t) = x(t) + R(t)p_0 \]
$R(t) =$

**x-axis** of the body coordinate system expressed in the body coordinate system is $(1, 0, 0)$

**y-axis** $\rightarrow (0, 1, 0)$

**z-axis** $\rightarrow (0, 0, 1)$.

Let $R(t) = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$.

$\Rightarrow R(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Observation: the first column of the rotation matrix $R(t)$ is the $x$-axis of the body coordinate system expressed in the world coordinate system.

$\therefore R(t) = \begin{vmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{zx} & r_{zy} & r_{zz} \end{vmatrix}$.
Linear Velocity:

\[ V(t) = \dot{x}(t) \]

Imagine that the orientation of the body is fixed.

Angular Velocity:

Imagine that we fixed the COM. Further assume that the body is spinning about an axis that passes through the COM.

We describe this spin as a vector \( \omega(t) \).

direction gives the direction of axis about which the body is rotating.
magnitude tells how fast the body is rotating (revolutions/time).

\[ \omega(t) \text{ angular velocity} \]

\[ V(t) \text{ linear velocity} \]
Relationship between $\mathbf{R}(t)$ and $\mathbf{w}(t)$.

Let's consider a vector $\mathbf{r}(t)$ that is rigidly attached to the rigid body.

$\mathbf{r}(t)$ is not affected by the translational effects.

What is the instantaneous velocity of TIP?

$|\mathbf{w}(t) \times \mathbf{b}| = |\mathbf{w}(t)||\mathbf{b}|$

$\mathbf{r}(t) = \mathbf{a} + \mathbf{b}$

We can write $\dot{\mathbf{r}}(t) = \mathbf{w}(t) \times \mathbf{r}(t)$

Let's apply this to $\mathbf{R}(t)$.

$$\dot{\mathbf{R}} = \begin{bmatrix} \mathbf{w}(t) \times \mathbf{r} & \mathbf{w}(t) \times \mathbf{r} & \mathbf{w}(t) \times \mathbf{r} \end{bmatrix}$$
Cross-Products.

Given a vector \( a = (a_x, a_y, a_z) \).

Define

\[
\mathbf{a}^* = \begin{bmatrix}
0 & -a_z & a_y \\
-a_z & 0 & -a_x \\
a_y & a_x & 0
\end{bmatrix},
\]

then \( \mathbf{a} \times \mathbf{b} = \mathbf{a}^* \mathbf{b} \).

Therefore

\[
\mathbf{R} = \begin{bmatrix}
\mathbf{w}(t)^* \\
\mathbf{h}(t)^*
\end{bmatrix} \begin{bmatrix}
\mathbf{h}(t) \\
\mathbf{w}(t)
\end{bmatrix} = \mathbf{w}(t)^* \mathbf{R}(t)
\]

matrix-matrix multiplication
Mass of a body.

Consider $N$ small particles: $m_1, \ldots, m_N$, each at position $r_{0i}$, $1 \leq i \leq N$.

Total mass of the body: $M = \sum_{i=1}^{N} m_i$.

Position of the $i^{th}$ particle:

$$r_i(t) = R(t) \times r_{0i} + x(t).$$

Velocity of the $i^{th}$ particle:

$$\dot{r}_i(t) = \omega(t) \times r_{0i} + \nu(t)$$

$$= \omega(t) \times (R(t) \times r_{0i} + x(t)) - x(t) + \nu(t)$$

$$= \omega(t) \times (r_i(t) - x(t)) + \nu(t)$$

The expression $\omega(t) \times (r_i(t) - x(t))$ represents the rotational (angular) component, and $\nu(t)$ represents the linear component.
Dynamics of a rigid body.

Goal: separate into linear and angular components.

Center of mass in the world space
\[ \sum m_i \mathbf{r}_i(t) = \frac{\sum m_i \mathbf{r}_i(t)}{M} \]

We often use the center of mass coordinate system. It means that in the body space:
\[ \sum_{i=1}^{n} m_i \mathbf{r}_i = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \sum_{i=1}^{n} m_i \mathbf{r}_i = 0 \]

We have been implicitly assuming \( x(t) \) to be the com of the body. Is it true?
\[ \sum m_i x_i(t) = \frac{\sum m_i (r_i(t) \mathbf{r}_i + x(t))}{M} \]
\[ = \frac{k(t) \sum m_i \mathbf{r}_i + \sum m_i x(t)}{M} \]
\[ = x(t) \frac{\sum m_i}{M} \]
\[ = x(t) \]
**Force and Torque.**

\[ T_i(t) = (r_i(t) - x(t)) \times F_i(t) \]

Differs from force. The torque acting on a particle also depends upon the distance of the particle from the center of mass.

**Total external force:** \( F(t) = \sum F_i(t) \)

**Total torque:** \( T(t) = \sum T_i(t) = \sum (r_i(t) - x(t)) \times F_i(t) \).
Linear Momentum.

\[ p = m v \]

Total linear momentum

\[ P(t) = \sum m_i \dot{x}_i(t) \]

\[ \dot{x}_i(t) = v_i(t) + \omega_i(t) \times (\lambda_i(t) - x(t)) \]

\[ P(t) = \sum m_i \left( v_i(t) + \omega_i(t) \times (\lambda_i(t) - x(t)) \right) \]

\[ = \sum m_i v_i(t) + \omega(t) \times \sum m_i (\lambda_i(t) - x(t)) \]

\[ \sum m_i (\lambda_i(t) - x(t)) \]

\[ = \lambda(t) \sum m_i \lambda_i \]

\[ = 0 \]

\[ P(t) = \sum m_i v_i(t) \]

\[ \Rightarrow v(t) = \frac{P(t)}{M} \]

\[ \Rightarrow \dot{v}(t) = \frac{P(t)}{M} = \frac{P(t)}{M} \quad \text{Newton's Second Law} \]

(😊)
Angular Momentum:

Defined by the equation

\[ \mathbf{L}(t) = \mathbf{I}(t) \mathbf{\omega}(t) \]

angular momentum

inertia tensor

a 3x3 matrix.

Also, \( \mathbf{I}(t) = \mathbf{\Omega}(t) \) describes the relationship between torque and inertia tensor.

\[ \mathbf{I}(t) = \sum m_i \begin{pmatrix} x_i^2 + z_i^2 & -x_i z_i & -y_i z_i \\ -x_i z_i & x_i^2 + z_i^2 & -y_i z_i \\ -y_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix} \]

\[ \mathbf{I}(t) = \mathbf{R}(t) \mathbf{I}_{\text{body}} \mathbf{R}(t)^T \]

defined in the body space so is constant.

Also, \( \mathbf{I}(t) = \mathbf{R}(t) \mathbf{I}_{\text{body}} \mathbf{R}(t)^T \)

\( \mathbf{R}(t)^T = \mathbf{R}(t)^{-1} \)

Also a constant during simulation.
Rigid Body Equations of Motion:

\[
Y(t) = \begin{pmatrix} x(t) \\ R(t) \\ P(t) \\ L(t) \end{pmatrix}
\]

\[
v(t) = \frac{p(t)}{M}, \quad I(t) = R(t) \mathbf{I}_{\text{body}} R(t)^T, \quad \omega(t) = I(t)^{-1} L(t)
\]

\[
\frac{dY(t)}{dt} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ P(t) \\ T(t) \end{pmatrix}
\]