Rigid Body Collisions

Consider two balls $A$ and $B$. Assume that the balls are rigid.

Radius of ball $A$: $r^A$
" B: $r^B$

Velocity of ball $A$: $v^A$
" B: $v^B$

Position of ball $A$: $x^A$
" B: $x^B$

Mass of ball $A$: $m^A$
" B: $m^B$

* There are no rotational effects.

Velocity of collision point $P$ for ball $A$: $v^{AP} = v^A$
" B: $v^{BP} = v^B$

Collision Detection
- compute normal $n = (p^A - p^B)/|p^A - p^B|$,
- compute relative velocity $v^{AB} = v^{AP} - v^{BP}$

$v^{AB} \cdot n = 0 \rightarrow$ resting contact
$< 0 \rightarrow$ collision imminent
$> 0 \rightarrow$ A and B moving away.
Collision Response

1. Use Newton’s Law of Restitution for instantaneous collision with no friction.

Impulse: an infinite force applied for a very short duration. Impulse is equal to the change in momentum.

\[ J = \Delta p = m v_1 - m v_2 \]

\[ \Rightarrow v_2 = v_1 - \frac{J}{m} \]

\( v_1 \) is the velocity of a body before collision and \( v_2 \) is the velocity of the body after collision.

no gravity

no friction

conservation of momentum.

\( J \) for first body is equal to \(-J\) of the second body.
(2) Empirical model of frictionless collisions.

\[ v_{2}^{AB}.n = -e v_{1}^{AB}.n \]

Relative velocity between the two bodies after collision (along the collision direction \(n\)) is a function of the relative velocity between the two bodies before collision.

\( e \) is called the Coefficient of Restitution.

- \( e = 1 \) elastic collision, no loss of kinetic energy.
- \( e = 0 \) perfectly inelastic, total loss of kinetic energy.
- \( 0 < e < 1 \) some loss of kinetic energy.

Given (1) and (2) above, let's solve for the velocities of balls A and B after collisions.

(refer pg. 1) \( v_{1}^{AP} \) = velocity of P in A, before collision.
\( v_{1}^{BP} \) = velocity of P in B before collision.
\[ v_{2A} = \text{velocity of A after collision} \]
\[ v_{2B} = \text{velocity of B after collision} \]

From (1):
\[ v_{2A} = v_{1A} + \frac{j n}{m_A} \quad (\text{A}) \]
\[ v_{2B} = v_{2A} - \frac{j n}{m_B} \quad (\text{B}) \]

Subtract (B) from (A):
\[ (v_{2A} - v_{2B}) = (v_{1A} - v_{2B}) + \left( \frac{1}{m_A} + \frac{1}{m_B} \right) j n \]

\[ \Rightarrow \quad v_{2}^{AB} = v_{1}^{AB} + \left( \frac{1}{m_A} + \frac{1}{m_B} \right) j n \quad (\text{C}) \]

From (2):
\[ v_{2}^{AB} \cdot n = -e v_{1}^{AB} \cdot n \quad (\text{D}) \]

Using (C) and (D):
\[ -e v_{1}^{AB} \cdot n = v_{1}^{AB} \cdot n + \left( \frac{1}{m_A} + \frac{1}{m_B} \right) j n \cdot n \]
\[ \frac{1}{m_A} + \frac{1}{m_B} \]
\[ \text{unit vectors} \]
\[ j = -(1+e)v_1^{AB} \cdot n \]
\[
\left( \frac{1}{m_A} + \frac{1}{m_B} \right)
\]

**Strategy**

You are given \( v_1^A, v_1^B, m_A, m_B, e, x^A, x^B \). Compute \( n \) and then compute \( j \).

\( j \) for \( A = -j \) for \( B \).

Compute \( v_2^A = v_1^A - \frac{jn}{m_A} \)

and \( v_2^B = v_1^B - \frac{jn}{m_B} \)
Rigid Body Collision with Rotational Effects.

Collision point \( P \) in \( A \): \( \mathbf{r}_{AP} \)

" \( B \): \( \mathbf{r}_{BP} \)

Collision will be felt along \( \mathbf{n} \)

Before Collision

Velocity of \( P \) in \( A \): \( \mathbf{v}_1^A + \omega_1^A \times \mathbf{r}_{AP} = \mathbf{v}_{1AP} \)

" \( B \): \( \mathbf{v}_1^B + \omega_1^B \times \mathbf{r}_{BP} = \mathbf{v}_{1BP} \)

we are interested in \( \mathbf{v}_2^A, \mathbf{v}_2^B, \omega_2^A \) and \( \omega_2^B \) i.e., velocities after collision.

Mass of body \( A \): \( m^A \) and inertia tensor: \( \mathbf{I}_A \) (world)

" \( B \): \( m^B \)
From (1) pg. 2

\[ v_{2A} = v_{1A} + \frac{jm}{mA} \quad \text{and for rotational component} \]
\[ w_{2A} = w_{1A} + (IA)^{-1} r_{AP} \times jn \]

Similarly for body B

\[ v_{2B} = v_{1B} - \frac{jm}{mB} \]
\[ w_{2B} = w_{1B} + (IB)^{-1} r_{BP} \times jn \]

We can use (1) and (2) to get

\[ j = \frac{-(1+e) V_{AB}}{\left(\frac{1}{mA} + \frac{1}{mB}\right) + n \cdot (IA)^{-1} (r_{AP} \times n) \times r_{AP} + n \cdot (IB)^{-1} (r_{BP} \times n) \times r_{BP}} \]

**Strategy.**

1. Compute \( n \).
2. Compute \( j \).
3. Compute \( v_{2A}, w_{2A}, v_{2B} \) and \( w_{2B} \).
4. Update the state of the two bodies.
5. Continue with simulation.